XVI Congreso de Confiabilidad
San Sebastián, 3 y 4 de diciembre de 2014
Failure modeling of an electrical $N$-component framework by the non-stationary Markovian arrival process

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XVI Congreso de Confiabilidad
San Sebastián, December 3rd, 2014
• Electrical components are essential in everyday operations and life and it is crucial that they do not fail.

• Reliability: the probability of a system or a component to function under stated conditions for a specified period of time.

• Failures can be caused by faults or errors in the components that comprise the system, or alternatively, the structure that comprises the component.

• As a failure occurs, a repair or replacement may take place in order that the component goes back to functioning as soon as possible.
Our Data

Component 1

\[ t_1^{(1)} \quad t_2^{(1)} \quad \ldots \quad t_{21}^{(1)} \]

Component 2

\[ t_1^{(2)} \quad t_2^{(2)} \quad \ldots \quad t_{16}^{(2)} \]

Component 926

\[ t_1^{(926)} \quad t_2^{(926)} \quad \ldots \quad t_{18}^{(926)} \]

Minimum number of failures=1.
Maximum number of failures=42.
The considered random variables are

\[ T_k = \{ t_k^{(1)}, t_k^{(2)}, t_k^{(3)}, \ldots, t_k^{(926)} \} \quad k = 1, 2, \ldots, 42. \]

The 926 components are considered to be equal, since the company states they are built with the same structure.
Our Data

- A total of 32 (out of 300) pairs \((T_k, T_l), k, l \in \{1, \ldots, 25\}, k < l\), presented a correlation coefficient ranging in \([0.25, 0.7194]\). In addition, 11 (out of 300) pairs had a correlation coefficient which ranged in \([-0.3266, -0.25]\).

  The \(T_k\)s are correlated

- A Kolmogorov-Smirnov (K-S) test rejected the equality in distribution for 52% pairs of the samples, which implies that the inter-failure times cannot be considered identically distributed nor independent.

  The \(T_k\)s are not identically distributed
Our Data

The $T_k$s are not exponential
Example 1: Danish fire insurance losses
Example 2: Software reliability data
The MAP

- Versatile Markovian point process (Neuts, 1979).
- Markovian Arrival process or MAP (Lucantoni et al. 1990).
  1. Stationary MAPs are dense in the family of stationary point processes.
  2. Tractability of the Poisson process.
  3. Dependent inter-failure times.
  4. Non-exponential inter-failure times.

- Special cases:
  1. Phase-type renewal processes (Erlang and Hyperexponential),
  2. Non-renewal processes as the Markov-modulated Poisson process (MMPP).
Definition of the 2-state MAP or MAP₂

- Continuous Markov chain \( J(t) \), state space \( S = \{1, 2\} \) and generator matrix \( D \).
- Initial state \( i_0 \in S \) given by an initial probability vector \( \alpha = (\alpha, 1 - \alpha) \).
- At the end of a sojourn time in state \( i \), exponentially distributed with parameter \( \lambda_i > 0 \), two possible transitions:
  1. With probability \( p_{ij1} \) the MAP enters state \( j \in S \) and a single arrival occurs.
  2. With probability \( p_{ij0} \) the MAP enters state \( j \) without arrivals, \( j \neq i \).
- The MAP₂ process is characterized by \( M = \{\alpha, \lambda, P₀, P₁\} \), where \( \lambda = (\lambda₁, \lambda₂) \), and

\[
P₀ = \begin{pmatrix} 0 & p_{120} \\ p_{210} & 0 \end{pmatrix}, \quad P₁ = \begin{pmatrix} p_{111} & p_{121} \\ p_{211} & p_{221} \end{pmatrix}
\]
Transition diagram: $MAP_2$
The $\text{MAP}_2$ process can also be characterized by the set $\mathcal{M} = \{\alpha, D_0, D_1\}$.

Rate matrices

$$D_0 = \begin{pmatrix} x & y \\ z & u \end{pmatrix}, \quad D_1 = \begin{pmatrix} w & -x - y - w \\ v & -z - u - v \end{pmatrix},$$

where

$$x = -\lambda_1, \quad y = \lambda_1 p_{120}, \quad w = \lambda_1 p_{111},$$

$$z = \lambda_2 p_{210}, \quad u = -\lambda_2, \quad v = \lambda_2 p_{211}.$$

$D \equiv D_0 + D_1$ is the generator of $J(t)$, with stationary probability vector denoted by $\pi$. 
Some Properties

- The stationary probability vector $\phi$ is calculated as
  \[ \phi P^* = \phi, \]
  where $P^*$ is the transition probability matrix, given by
  \[ P^* = (-D_0)^{-1} D_1. \]

- The CDF and moments of $\{ T_k \}_{k=1,2,...,42}$ are given by,
  \[ F_{T_k}(t) = 1 - \alpha_k e^{D_0 t} e. \]
  \[ \mu_{k,m} = E(T_k^m) = m!\alpha_k (-D_0)^{-m} e. \]
  where, $\alpha_k = \alpha (P^*)^{k-1}$ and $T_k \sim PH \{\alpha_k, D_0\}$. 
Some properties

Concerning the counting process \( \{N(t), \ t \geq 0\} \)

- The probability of \( n \) failures at time \( t \) is given by,

\[
P(N(t) = n \mid N(0) = 0) = \alpha P(n, t)e,
\]

where the probability of \( n \) failures in the interval \((0, t] \) is given by the matrix \( P(n, t) \).

- The expected number of failures at time \( t \), \( E(N(t) \mid N(0) = 0) \), is computed from,

\[
M_1(t) = \sum_{n=0}^{\infty}nP(n, t).
\]
Rodríguez et al. (2014) defined the canonic representation of the non-stationary $MAP_2$ in terms of the eigenvalue different from zero of $P^*$, defined $\gamma$. So, if $\gamma > 0$, then

$$\tilde{\alpha} = (\tilde{\alpha}, 1 - \tilde{\alpha}), \quad \tilde{D}_0 = \begin{pmatrix} \tilde{x} & \tilde{y} \\ 0 & \tilde{u} \end{pmatrix}, \quad \tilde{D}_1 = \begin{pmatrix} -\tilde{x} - \tilde{y} & 0 \\ \tilde{v} & -\tilde{u} - \tilde{v} \end{pmatrix},$$

On the contrary, if $\gamma \leq 0$, the canonical representation is given by

$$\tilde{\alpha} = (\tilde{\alpha}, 1 - \tilde{\alpha}), \quad \tilde{D}_0 = \begin{pmatrix} \tilde{x} & \tilde{y} \\ 0 & \tilde{u} \end{pmatrix}, \quad \tilde{D}_1 = \begin{pmatrix} 0 & -\tilde{x} - \tilde{y} \\ -\tilde{u} - \tilde{v} & \tilde{v} \end{pmatrix},$$

where $\tilde{u} \leq \tilde{x} < 0$, $\tilde{x} + \tilde{y} \leq 0$ and $\tilde{u} + \tilde{v} \leq 0$.

The stationary version of the $MAP_2$ is obtained by setting $\alpha = \phi$. 
Non-Stationary vs. Stationary version

- In the stationary version, the probability vector is the stationary probability distribution $\phi$, we have that
  \[ P(X_n = i) = \phi(i). \]

  $\implies$ $T_k$ are identically distributed
  \[ T_k \sim PH\{\phi, D_0\}. \]

- In the non-stationary version, the probability vector is arbitrary, $\alpha$, and
  \[ P(X_j = i) = \left[ \alpha(P^\ast)^{j-1} \right](i), \quad \text{for } 1 \leq j \leq n. \]

  $\implies$ $T_k$ are not identically distributed.
  \[ T_k \sim PH\{\alpha, D_0\}. \]

  In particular,
  \[ \lim_{n \to \infty} \alpha(P^\ast)^n = \phi. \]
A number of articles have considered statistical estimation for the MAPs, but always under the assumption that the process is in its stationary version, for example:

- Breuer (2002), Klemm et al. (2003) and Okamura et al. (2009), studied the inference for the MAP via the EM (Expectation-Maximization) algorithm.

- Bayesian inference for the MAP$_2$ has been studied by Ramírez-Cobo et al. (2013), where different algorithms are proposed.
We have $N$ real sequences of the operational times, $t^{(1)}, \ldots, t^{(N)}$ as observations, where

$$
t^{(1)} = \left( t^{(1)}_1, t^{(1)}_2, \ldots, t^{(1)}_{n_1} \right),
$$

$$
t^{(2)} = \left( t^{(2)}_1, t^{(2)}_2, \ldots, t^{(2)}_{n_2} \right),
$$

$$
\vdots
$$

$$
t^{(N)} = \left( t^{(N)}_1, t^{(N)}_2, \ldots, t^{(N)}_{n_N} \right),
$$

$n_i$ denotes the size of the sample $t^{(i)}$, for $i = 1, \ldots, N.$
Data & parameters of the model

- We assume that the $N$ components are identical and the sequences of operational times $t^{(1)}, \ldots, t^{(N)}$, are independent among them.

- The goal is to estimate the model parameters $\{\tilde{\alpha}, \tilde{D_0}, \tilde{D_1}\}$, i.e. $\{\tilde{\alpha}, \tilde{x}, \tilde{y}, \tilde{u}, \tilde{v}\}$, from the sample $\{t^{(1)}, t^{(2)}, \ldots, t^{(N)}\}$.

- Unlike classical model assumptions, we cannot assume that the random variables $\{T_k\}_{k \geq 1}$ are uncorrelated, and then, they cannot be considered independent. Also, the random variables $\{T_k\}_{k \geq 1}$ are not necessarily identically distributed.
Moment Matching method

We define a moment matching estimation approach where the population moments $\mu_{k,m}$ are matched by their empirical counterparts $\overline{\mu_{k,m}}$, computed as

$$\overline{\mu_{k,m}} = \frac{1}{N} \sum_{i=1}^{N} \left( t_k^{(i)} \right)^m .$$

This leads to solve the nonlinear system of equations defined by

$$\mu_{1,m} \left( \bar{\alpha}, \bar{x}, \bar{y}, \bar{u}, \bar{v} \right) = \overline{\mu_{1,m}}, \ \ m = 1, 2, 3,$$

$$\mu_{k,1} \left( \bar{\alpha}, \bar{x}, \bar{y}, \bar{u}, \bar{v} \right) = \overline{\mu_{k,1}}, \ \ k = 2, 3.$$
The previous system of equations may not have a feasible solution, therefore, we follow Carrizosa and Ramírez (2013), and seek instead the parameters \( \{\tilde{\alpha}, \tilde{x}, \tilde{y}, \tilde{u}, \tilde{v}\} \) that fulfills as much as possible those equalities, by means of the following optimization problem.

\[
\begin{align*}
\min & \quad \delta_T (\tilde{\alpha}, \tilde{D}_0, \tilde{D}_1) \\
\text{s.t.} & \quad \tilde{x}, \tilde{u} \leq 0, \\
& \quad \tilde{y}, \tilde{v} \geq 0, \\
& \quad -\tilde{x} - \tilde{y} \geq 0, \\
& \quad -\tilde{u} - \tilde{v} \geq 0, \\
& \quad 0 \leq \tilde{\alpha} \leq 1,
\end{align*}
\]
Moment Matching method

where the objective function is given by,

\[ \delta_\tau \left( \tilde{\alpha}, \tilde{D}_0, \tilde{D}_1 \right) = \tau \left\{ \left( \frac{r_1 \left( \tilde{\alpha}, \tilde{D}_0, \tilde{D}_1 \right) - \bar{r}_1}{\bar{r}_1} \right)^2 + \left( \frac{r_2 \left( \tilde{\alpha}, \tilde{D}_0, \tilde{D}_1 \right) - \bar{r}_2}{\bar{r}_2} \right)^2 \right. \\
+ \left. \left( \frac{r_3 \left( \tilde{\alpha}, \tilde{D}_0, \tilde{D}_1 \right) - \bar{r}_3}{\bar{r}_3} \right)^2 + \left( \frac{\mu_2 \left( \tilde{\alpha}, \tilde{D}_0, \tilde{D}_1 \right) - \bar{\mu}_2}{\bar{\mu}_2} \right)^2 \right\} \]

\( \tau \) is a penalty parameter that needs to be tuned, but setting \( \tau = 1 \) performs well in practice.
The optimization problem \((P)\) is solved by using the local search MATLAB’s routine \texttt{fmincon}\ (Optimization toolbox).

We perform a multistart approach (200 different starting points randomly selected are used) and keep the solution with minimum objective function \(\delta_T\).
Given the sample $t^{(1)}, \ldots, t^{(N)}$, problem ($P$) needs to be solved twice, one per each of the two canonical representations.

The estimated parameters under the model with highest log-likelihood are selected, where the log-likelihood of the sample is given by

$$\log f(t^{(1)}, \ldots, t^{(N)}|D_0, D_1) = \sum_{i=1}^{N} \log f(t^{(i)}|D_0, D_1).$$
Illustration with a real data set

We have the failure times of $N = 926$ electrical components, the length of the failure times is different for each component.

Components with less than 3 observations will not be considered. And samples of length larger than 30 will be considered.

The sample moments are given by

$$(\hat{\mu}_{1,1}, \hat{\mu}_{1,2}, \hat{\mu}_{1,3}, \hat{\mu}_{2,1}, \hat{\mu}_{3,1}) = (79.226, 7.478 \times 10^3, 7.2911 \times 10^3, 69.0582, 67.5977).$$

**First canonical form estimate:**

$$\hat{\alpha}^1 = (0.4608, 0.5392), \quad \hat{D}_0^1 = \begin{pmatrix} -0.2394 & 0.1345 \\ 0 & -0.0104 \end{pmatrix}, \quad \hat{D}_1^1 = \begin{pmatrix} 0.1049 & 0 \\ 0.0067 & 0.0037 \end{pmatrix},$$

with estimated moments given by

$$(\hat{\mu}_{1,1}, \hat{\mu}_{1,2}, \hat{\mu}_{1,3}, \hat{\mu}_{2,1}, \hat{\mu}_{3,1}) = (78.8950, 7.535 \times 10^3, 7.2633 \times 10^3, 69.0864, 67.5712),$$

and objective function equal to $\delta_1^1 = 9.0324 \times 10^{-5}$. 
Illustration with a real data set

Second canonical form estimate:

\[ \hat{\alpha}^2 = (0.8207, 0.1793), \quad \hat{D}_0^2 = \begin{pmatrix} -0.0104 & 0.0104 \\ 0 & -16.5378 \end{pmatrix}, \quad \hat{D}_1^2 = \begin{pmatrix} 0 & 0 \\ 11.7651 & 4.7727 \end{pmatrix}, \]

with estimated moments given by

\( (\hat{\mu}_{1,1}, \hat{\mu}_{1,2}, \hat{\mu}_{1,3}, \hat{\mu}_{2,1}, \hat{\mu}_{3,1}) = (78.7930, 7.5583 \times 10^3, 7.2513 \times 10^3, 68.3119, 68.3119), \)

and objective function \( \delta^2 = 4.0324 \times 10^{-4}. \)

The log-likelihoods are

\[ \log f(t^{(1)}, \ldots, t^{(N)}|\hat{D}_0^1, \hat{D}_1^1) = -5.3790 \times 10^4, \]

\[ \log f(t^{(1)}, \ldots, t^{(N)}|\hat{D}_0^2, \hat{D}_1^2) = -5.7335 \times 10^4, \]

which provides evidence in favor of the estimate \( \{\hat{\alpha}^1, \hat{D}_0^1, \hat{D}_1^1\}. \)
Estimated CDF vs. Empirical CDF

(p-value = 0.2671)

(p-value = 0.7867)

(p-value = 0.0014)
Counting process descriptors

Probabilities $P(N(t) = n)$ for $n \in \mathbb{N}$ and $t > 0$.

Expected number of failures at time $t$.

Probabilities $P(N(t) = n)$ for $n = 1, 2, 3, 4, 5$ and $t > 0$. 
Conclusions & Extensions

- The failure times are considered to be dependent and not identically distributed, an assumption which is realistic in practice.

- The canonical representation of the non-stationary version of the $MAP_2$ is considered to model the failure times.

- We present a moments matching method estimation procedure to fit the non-stationary second-order $MAP$ to sequences of operational times of $N$ electrical components that are structurally equal.

- From the estimated parameters of the model, a number of key performance measures regarding the counting process, as the probability of $N$ failures or the expected number of failures at time $t$, are inferred.
References