ASOCIACIÓN ESPAÑOLA PARA LA CALIDAD

XVI Congreso de Confiabilidad

San Sebastián, 3 y 4 de diciembre de 2014

Failure modeling of an electrical N-component framework by the non-stationary Markovian arrival process

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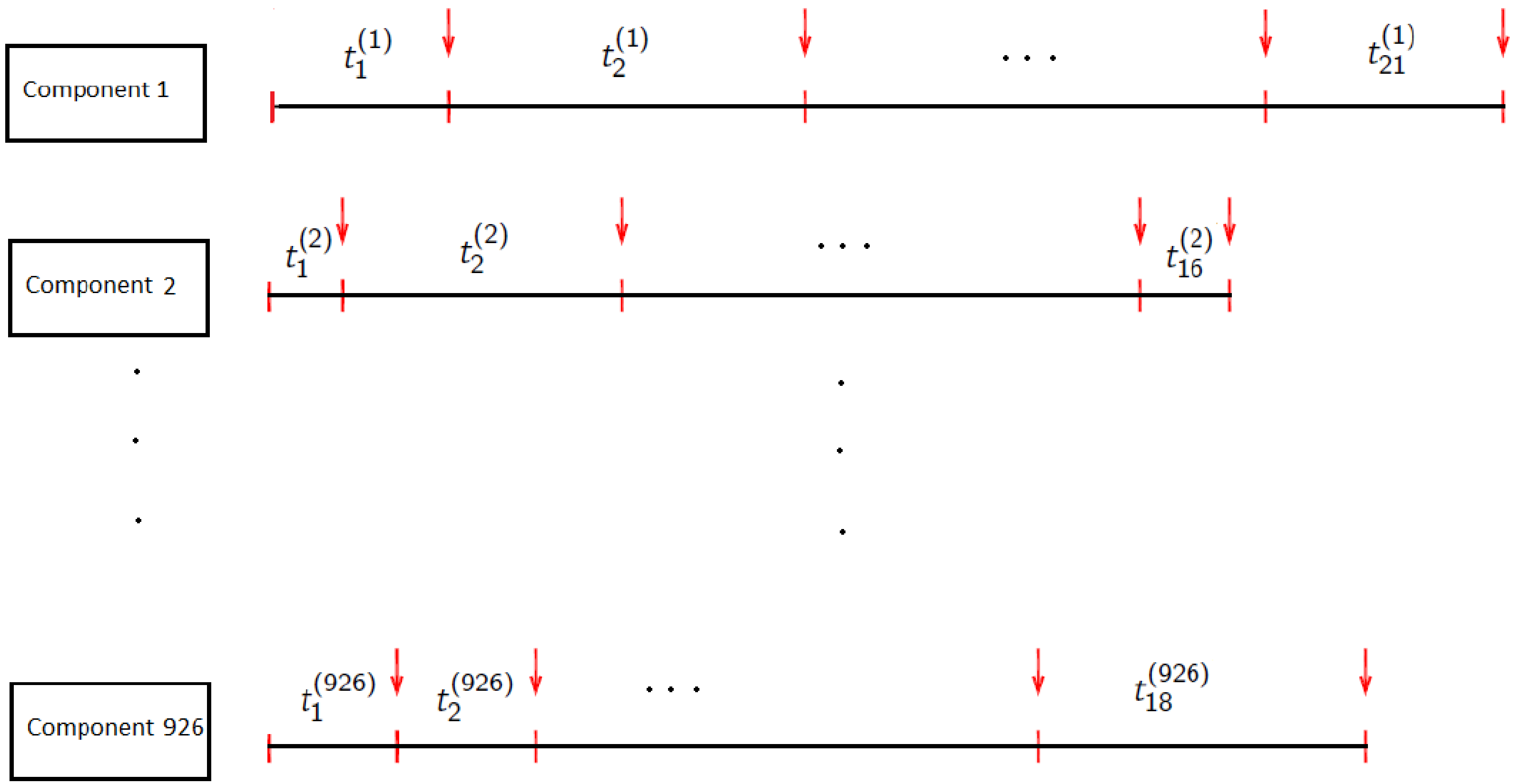
XVI Congreso de Confiabilidad
San Sebastián, December 3rd, 2014

The problem to be solved

- Electrical components are essential in everyday operations and life and it is crucial that they do not fail.
- Reliability: the probability of a system or a component to function under stated conditions for a specified period of time.
- Failures can be caused by faults or errors in the components that comprise the system, or alternatively, the structure that comprises the component.
- As a failure occurs, a repair or replacement may take place in order that the component goes back to functioning as soon as possible.

Our Data



Minimum number of failures=1.
Maximum number of failures=42.

Our Data

- The considered random variables are

$$T_k = \{t_k^{(1)}, t_k^{(2)}, t_k^{(3)}, \ldots, t_k^{(926)}\} \quad k = 1, 2, \ldots, 42.$$

- The 926 components are considered to be equal, since the company states they are built with the same structure.

Our Data

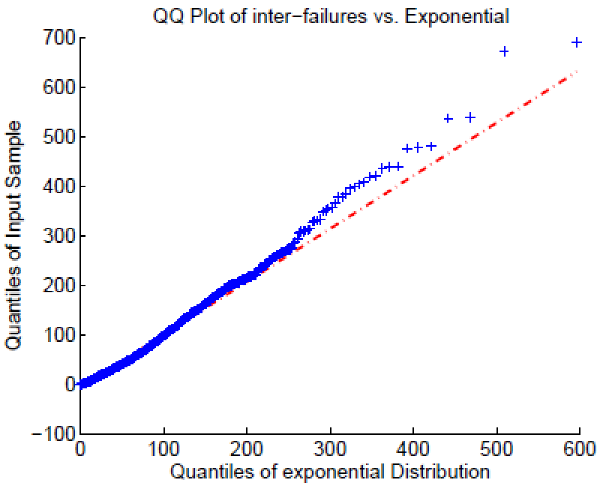
- A total of 32 (out of 300) pairs (T_k, T_l), $k, l \in \{1, \dots, 25\}$, $k < l$, presented a correlation coefficient ranging in $[0.25, 0.7194]$. In addition, 11 (out of 300) pairs had a correlation coefficient which ranged in $[-0.3266, -0.25]$.

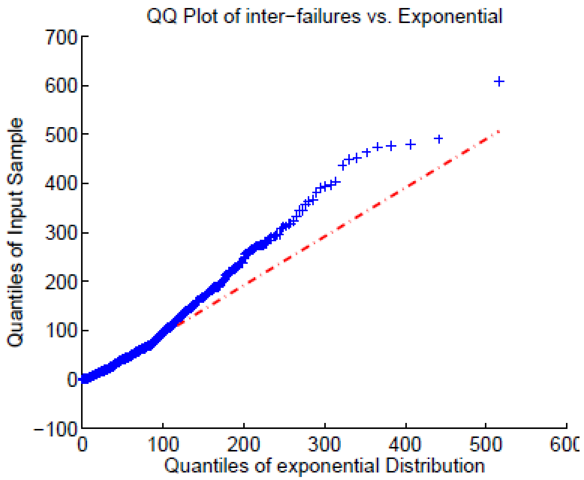
The T_ks are correlated

- A Kolmogorov-Smirnov (K-S) test rejected the equality in distribution for 52% pairs of the samples, which implies that the inter-failure times cannot be consider identically distributed nor independent.

The T_ks are not identically distributed

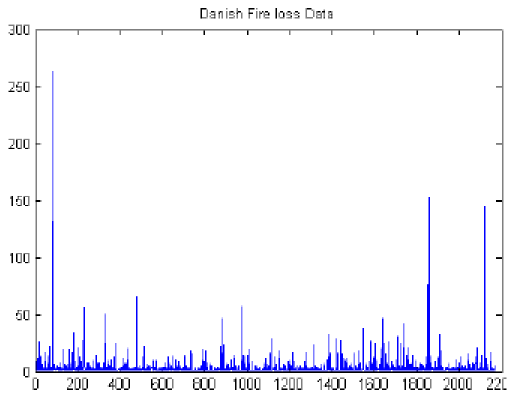
Our Data

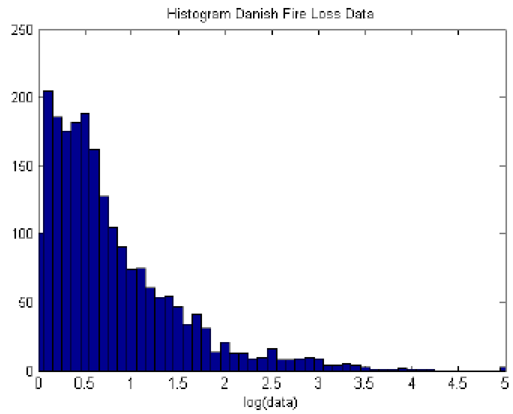


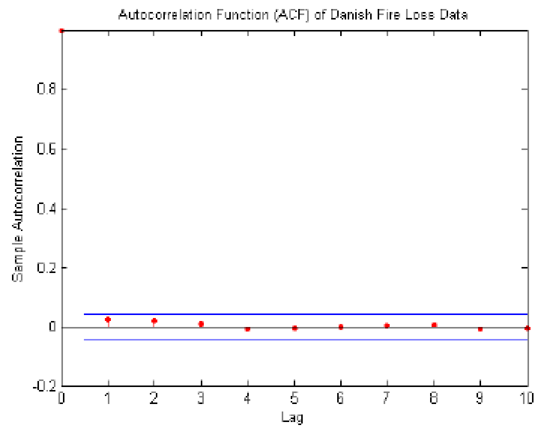


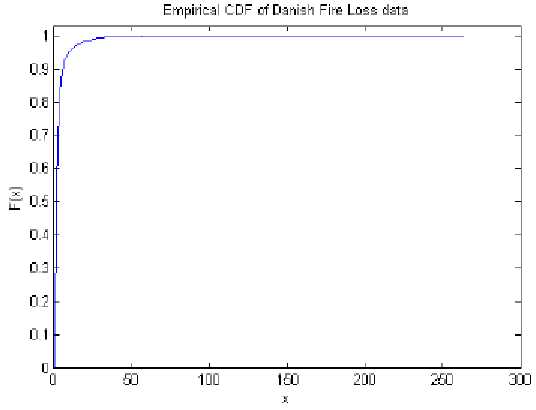
The T_ks are not exponential

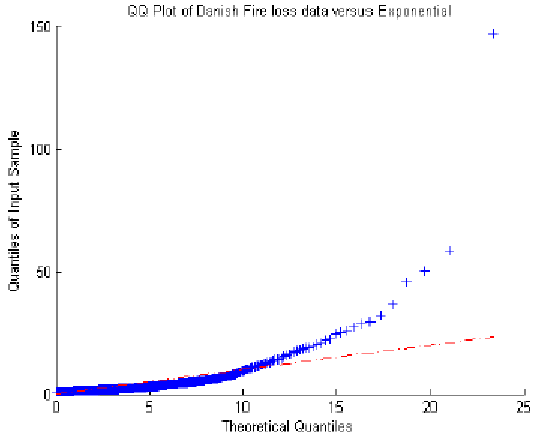
Example 1: Danish fire insurance losses



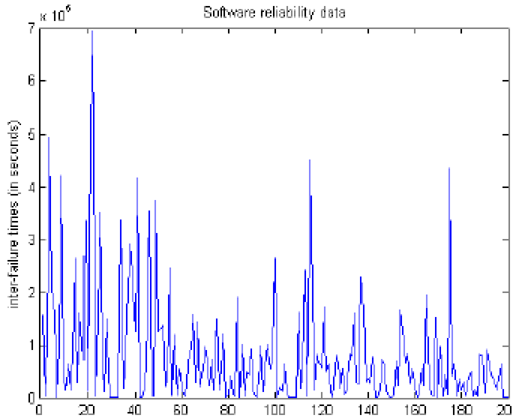


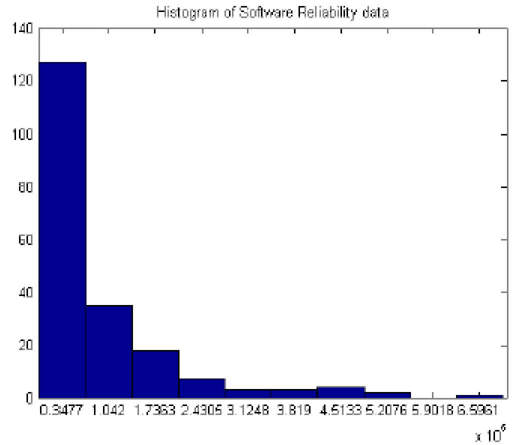


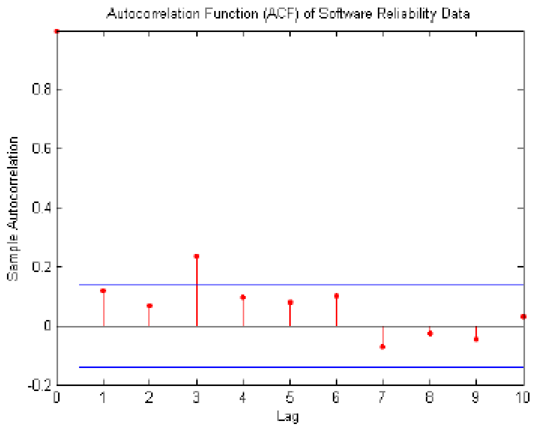


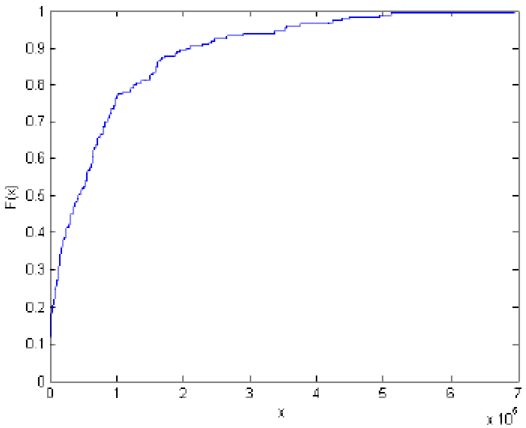


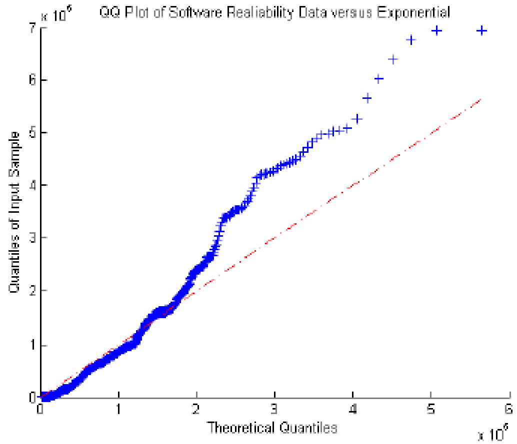
Example 2: Software reliability data











The *MAP*

- *Versatile Markovian point process* (Neuts, 1979).
- *Markovian Arrival process* or *MAP* (Lucantoni et al. 1990).
  1. Stationary *MAP*s are **dense** in the family of stationary point processes.
  2. **Tractability** of the Poisson process.
  3. **Dependent** inter-failure times.
  4. **Non-exponential** inter-failure times.
- Special cases:
  1. Phase-type renewal processes (Erlang and Hyperexponential),
  2. Non-renewal processes as the Markov-modulated Poisson process (*MMPP*).

Definition of the 2-state MAP or MAP_2

- Continuous Markov chain $J(t)$, state space $\mathcal{S} = \{1, 2\}$ and generator matrix D.

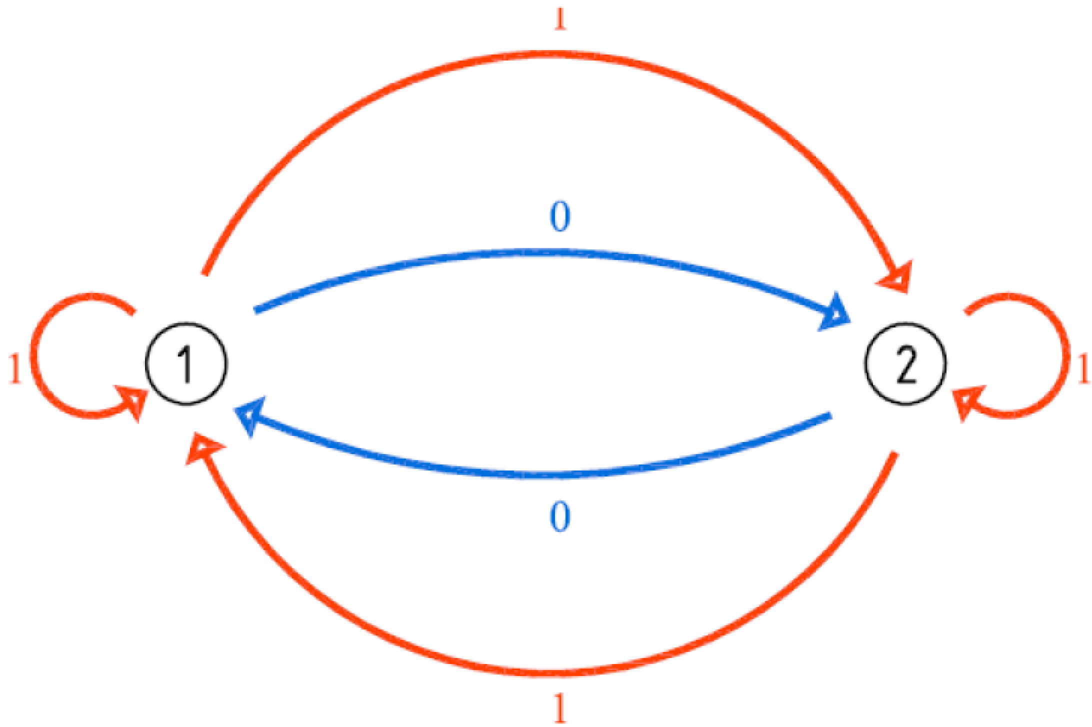
- Initial state $i_0 \in \mathcal{S}$ given by an initial probability $\boldsymbol{\alpha} = (\alpha, 1-\alpha)$.

- At the end of a sojourn time in state i, exponentially distributed with parameter $\lambda_i > 0$, two possible transitions:
  1. With probability p_{ij1} the MAP enters state $j \in \mathcal{S}$ and a **single arrival** occurs.
  2. With probability p_{ij0} the MAP enters state j **without arrivals**, $j \neq i$

- The MAP_2 process is characterized by $\mathcal{M} = \{\boldsymbol{\alpha}, \boldsymbol{\lambda}, P_0, P_1\}$, where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$, and

$$P_0 = \begin{pmatrix} 0 & p_{120} \\ p_{210} & 0 \end{pmatrix}, \qquad P_1 = \begin{pmatrix} p_{111} & p_{121} \\ p_{211} & p_{221} \end{pmatrix}$$

Transition diagram: MAP_2



Alternative characterization

- The MAP_2 process can also be characterized by the set $\mathcal{M} = \{\alpha, D_0, D_1\}$.

- Rate matrices

$$D_0 = \begin{pmatrix} x & y \\ z & u \end{pmatrix}, \quad D_1 = \begin{pmatrix} w & -x-y-w \\ v & -z-u-v \end{pmatrix},$$

where

$$x = -\lambda_1, \quad y = \lambda_1 p_{120}, \quad w = \lambda_1 p_{111},$$
$$z = \lambda_2 p_{210}, \quad u = -\lambda_2, \quad v = \lambda_2 p_{211}.$$

- $D \equiv D_0 + D_1$ is the generator of $J(t)$, with stationary probability vector denoted by $\boldsymbol{\pi}$.

Some Properties

- The stationary probability vector ϕ is calculated as

$$\phi P^{\star} = \phi,$$

where $P^{\star}$ is the transition probability matrix, given by $P^{\star} = (-D_0)^{-1} D_1$.

- The CDF and moments of $\{T_k\}_{k=1,2,\ldots,42}$ are given by,

$$F_{T_k}(t) = 1 - \boldsymbol{\alpha}_k e^{D_0 t}\mathbf{e}.$$

$$\mu_{k,m} = E\left(T_k^m\right) = m!\boldsymbol{\alpha}_k \left(-D_0\right)^{-m}\mathbf{e},$$

where, $\boldsymbol{\alpha}_k = \boldsymbol{\alpha}\left(P^{\star}\right)^{k-1}$ and $T_k \sim PH\{\boldsymbol{\alpha}_k, D_0\}$.

Some properties

Concerning the counting process $\{N(t),\ t \geq 0\}$

- The probability of n failures at time t is given by,

$$P\left(N(t) = n \mid N(0) = 0\right) = \boldsymbol{\alpha} P(n,t)\mathbf{e},$$

  where the probability of n failures in the interval $(0,t]$ is given by the matrix $P(n,t)$.

- The expected number of failures at time t, $E\left(N(t) \mid N(0) = 0\right)$, is computed from,

$$M_1(t) = \sum_{n=0}^{\infty} nP(n,t).$$

Canonical Representation

Rodríguez et al. (2014) defined the canonic representation of the non-stationary MAP_2 in terms of the eigenvalue different from zero of $P^\star$, defined γ. So, if $\gamma > 0$, then

$$\widetilde{\boldsymbol{\alpha}} = (\widetilde{\alpha},\ 1-\widetilde{\alpha}), \quad \widetilde{D}_0 = \begin{pmatrix} \tilde{x} & \tilde{y} \\ 0 & \tilde{u} \end{pmatrix}, \quad \widetilde{D}_1 = \begin{pmatrix} -\tilde{x}-\tilde{y} & 0 \\ \tilde{v} & -\tilde{u}-\tilde{v} \end{pmatrix},$$

On the contrary, if $\gamma \leq 0$, the canonical representation is given by

$$\widetilde{\boldsymbol{\alpha}} = (\widetilde{\alpha},\ 1-\widetilde{\alpha}), \quad \widetilde{D}_0 = \begin{pmatrix} \tilde{x} & \tilde{y} \\ 0 & \tilde{u} \end{pmatrix}, \quad \widetilde{D}_1 = \begin{pmatrix} 0 & -\tilde{x}-\tilde{y} \\ -\tilde{u}-\tilde{v} & \tilde{v} \end{pmatrix},$$

where $\tilde{u} \leq \tilde{x} < 0$, $\tilde{x}+\tilde{y} \leq 0$ and $\tilde{u}+\tilde{v} \leq 0$.

The stationary version of the MAP_2 is obtained by setting $\boldsymbol{\alpha} = \boldsymbol{\phi}$.

Non-Stationary vs. Stationary version

- In the stationary version, the probability vector is the stationary probability distribution ϕ, we have that

$$P(X_n = i) = \phi(i),$$

$\Longrightarrow T_k$ **are identically distributed**

$$T_k \sim PH\{\phi, D_0\}.$$

- In the non-stationary version, the probability vector is arbitrary, α, and

$$P(X_j = i) = \left[\alpha(P^\star)^{(j-1)}\right](i), \quad \text{for } 1 \leq j \leq n.$$

$\Longrightarrow T_k$ **are not identically distributed.**

$$T_k \sim PH\{\alpha_k, D_0\}.$$

In particular,

$$\lim_{n \to \infty} \alpha(P^\star)^n = \phi.$$

Statistical Estimation

A number of articles have considered statistical estimation for the *MAP*s, but always under the assumption that the process is in its stationary version, for example:

- Breuer (2002), Klemm *et al.* (2003) and Okamura *et al.* (2009), studied the inference for the *MAP* via the EM (Expectation-Maximization) algorithm.
- Bayesian inference for the MAP_2 has been studied by Ramírez-Cobo *et al.* (2013), where different algorithms are proposed.

Data & parameters of the model

We have N real sequences of the operational times, $\mathbf{t}^{(1)}, \ldots, \mathbf{t}^{(N)}$ as observations, where

$$\begin{aligned}
\mathbf{t}^{(1)} &= \left(t_1^{(1)}, t_2^{(1)}, \ldots, t_{n_1}^{(1)}\right), \\
\mathbf{t}^{(2)} &= \left(t_1^{(2)}, t_2^{(2)}, \ldots, t_{n_2}^{(2)}\right), \\
&\vdots \\
\mathbf{t}^{(N)} &= \left(t_1^{(N)}, t_2^{(N)}, \ldots, t_{n_N}^{(N)}\right),
\end{aligned}$$

n_i denotes the size of the sample $\mathbf{t}^{(i)}$, for $i = 1, \ldots, N$.

Data & parameters of the model

- We assume that the N components are identical and the sequences of operational times $\mathbf{t}^{(1)}, \ldots, \mathbf{t}^{(N)}$, are independent among them.

- The goal is to estimate the model parameters $\{\widetilde{\boldsymbol{\alpha}}, \widetilde{D_0}, \widetilde{D_1}\}$, i.e. $\{\widetilde{\alpha}, \tilde{x}, \tilde{y}, \tilde{u}, \tilde{v}\}$, from the sample $\{\mathbf{t}^{(1)}, \mathbf{t}^{(2)}, \ldots, \mathbf{t}^{(N)}\}$.

- Unlike classical model assumptions, we cannot assume that the random variables $\{T_k\}_{k\geq 1}$ are uncorrelated, and then, they cannot be considered independent. Also, the random variables $\{T_k\}_{k\geq 1}$ are not necessarily identically distributed.

Moment Matching method

We define a moment matching estimation approach where the population moments $\mu_{k,m}$ are matched by their empirical counterparts $\overline{\mu_{k,m}}$, computed as

$$\overline{\mu_{k,m}} = \frac{1}{N} \sum_{i=1}^{N} \left(t_k^{(i)} \right)^m .$$

This leads to solve the nonlinear system of equations defined by

$$\begin{aligned} \mu_{1,m}\left(\widetilde{\alpha}, \tilde{x}, \tilde{y}, \tilde{u}, \tilde{v}\right) &= \overline{\mu_{1,m}}, \; m = 1, 2, 3, \\ \mu_{k,1}\left(\widetilde{\alpha}, \tilde{x}, \tilde{y}, \tilde{u}, \tilde{v}\right) &= \overline{\mu_{k,1}}, \; k = 2, 3. \end{aligned}$$

Moment Matching method

The previous system of equations may not have a feasible solution, therefore, ,we follow Carrizosa and Ramírez (2013), and seek instead the parameters $\{\widetilde{\alpha}, \tilde{x}, \tilde{y}, \tilde{u}, \tilde{v}\}$ that fulfills as much as possible those equalities, by means of the following optimization problem.

$$(P) \begin{cases} min & \delta_{\tau}\left(\widetilde{\alpha}, \widetilde{D}_0, \widetilde{D}_1\right) \\ s.t. & \tilde{x}, \tilde{u} \leq 0, \\ & \tilde{y}, \tilde{v} \geq 0, \\ & -\tilde{x} - \tilde{y} \geq 0, \\ & -\tilde{u} - \tilde{v} \geq 0, \\ & 0 \leq \tilde{\alpha} \leq 1, \end{cases}$$

Moment Matching method

where the objective function is given by,

$$
\begin{aligned}
\delta_\tau\left(\widetilde{\alpha}, \widetilde{D}_0, \widetilde{D}_1\right) \quad = \quad & \tau\left\{\left(\frac{r_1\left(\widetilde{\alpha}, \widetilde{D}_0, \widetilde{D}_1\right)-\bar{r}_1}{\bar{r}_1}\right)^2+\left(\frac{r_2\left(\widetilde{\alpha}, \widetilde{D}_0, \widetilde{D}_1\right)-\bar{r}_2}{\bar{r}_2}\right)^2\right. \\
& +\left(\frac{r_3\left(\widetilde{\alpha}, \widetilde{D}_0, \widetilde{D}_1\right)-\bar{r}_3}{\bar{r}_3}\right)^2+\left(\frac{\mu_2\left(\widetilde{\alpha}, \widetilde{D}_0, \widetilde{D}_1\right)-\bar{\mu}_2}{\bar{\mu}_2}\right)^2 \\
& \left.+\left(\frac{\mu_3\left(\widetilde{\alpha}, \widetilde{D}_0, \widetilde{D}_1\right)-\bar{\mu}_3}{\bar{\mu}_3}\right)^2\right\}
\end{aligned}
$$

τ is a penalty parameter that needs to be tuned, but setting $\tau = 1$ performs well in practice.

Solution to (P)

- The optimization problem (P) is solved by using the local search MATLAB's routine **fmincon** (Optimization toolbox).
- We perform a multistart approach (200 different starting points randomly selected are used) and keep the solution with minimum objective function δ_τ.

Select a canonical form

- Given the sample $\mathbf{t}^{(1)}, \ldots, \mathbf{t}^{(N)}$, problem (P) needs to be solved twice, one per each of the two canonical representations.

- The estimated parameters under the model with highest log-likelihood are selected, where the log-likelihood of the sample is given by

$$\log f(\mathbf{t}^{(1)}, \ldots, \mathbf{t}^{(N)} | D_0, D_1) = \sum_{i=1}^{N} \log f(\mathbf{t}^{(i)} | D_0, D_1).$$

Illustration with a real data set

We have the failure times of $N = 926$ electrical components, the length of the failure times is different for each component.

Components with less than 3 observations will not be considered. And samples of length larger than 30 will be considered.

The sample moments are given by

$$(\overline{\mu_{1,1}},\ \overline{\mu_{1,2}},\ \overline{\mu_{1,3}},\ \overline{\mu_{2,1}},\ \overline{\mu_{3,1}}) = (79.226, 7.478\times10^3, 7.2911\times10^3, 69.0582, 67.5977).$$

First canonical form estimate:

$$\hat{\alpha}^1 = (0.4608, 0.5392), \quad \hat{D}_0^1 = \begin{pmatrix} -0.2394 & 0.1345 \\ 0 & -0.0104 \end{pmatrix}, \quad \hat{D}_1^1 = \begin{pmatrix} 0.1049 & 0 \\ 0.0067 & 0.0037 \end{pmatrix},$$

with estimated moments given by

$$(\widehat{\mu_{1,1}},\ \widehat{\mu_{1,2}},\ \widehat{\mu_{1,3}},\ \widehat{\mu_{2,1}},\ \widehat{\mu_{3,1}}) = (78.8950, 7.535\times10^3, 7.2633\times10^3, 69.0864, 67.5712),$$

and objective function equal to $\delta_\tau^1 = 9.0324 \times 10^{-5}$.

Illustration with a real data set

Second canonical form estimate:

$$\hat{\alpha}^2 = (0.8207, 0.1793), \quad \hat{D}_0^2 = \begin{pmatrix} -0.0104 & 0.0104 \\ 0 & -16.5378 \end{pmatrix}, \quad \hat{D}_1^2 = \begin{pmatrix} 0 & 0 \\ 11.7651 & 4.7727 \end{pmatrix},$$

with estimated moments given by

$$(\widehat{\mu_{1,1}},\ \widehat{\mu_{1,2}},\ \widehat{\mu_{1,3}},\ \widehat{\mu_{2,1}},\ \widehat{\mu_{3,1}}) = (78.7930, 7.5583\times10^3, 7.2513\times10^3, 68.3119, 68.3119),$$

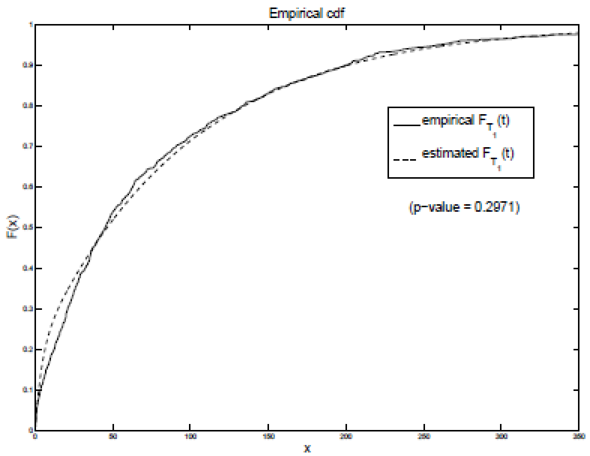
and objective function $\delta_\tau^2 = 4.0324 \times 10^{-4}$.
The log-likelihoods are

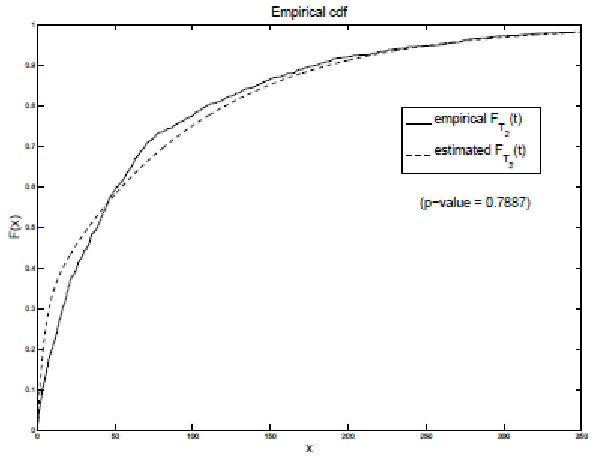
$$\log f(\mathbf{t}^{(1)}, \ldots, \mathbf{t}^{(N)} | \hat{D}_0^1, \hat{D}_1^1) = -5.3790 \times 10^4,$$

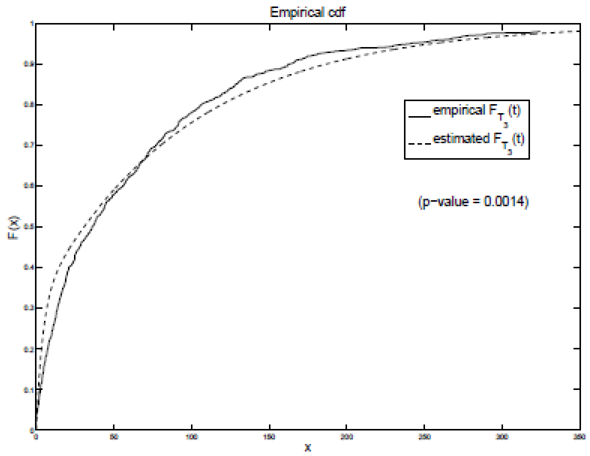
$$\log f(\mathbf{t}^{(1)}, \ldots, \mathbf{t}^{(N)} | \hat{D}_0^2, \hat{D}_1^2) = -5.7335 \times 10^4,$$

which provides evidence in favor of the estimate $\{\hat{\alpha}^1, \hat{D}_0^1, \hat{D}_1^1\}$.

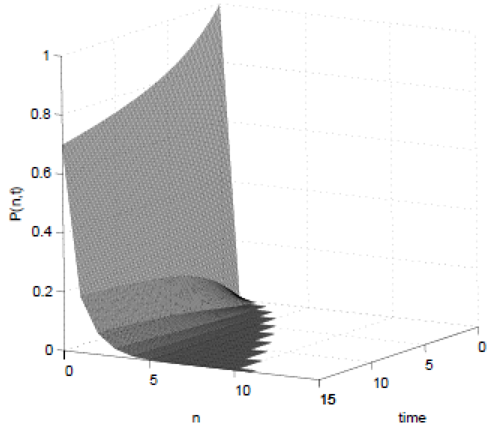
Estimated CDF vs. Empirical CDF



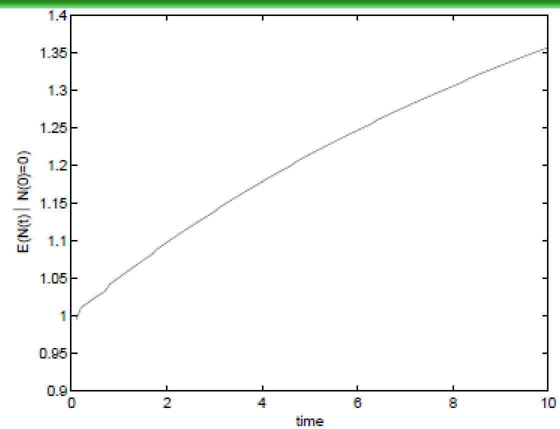




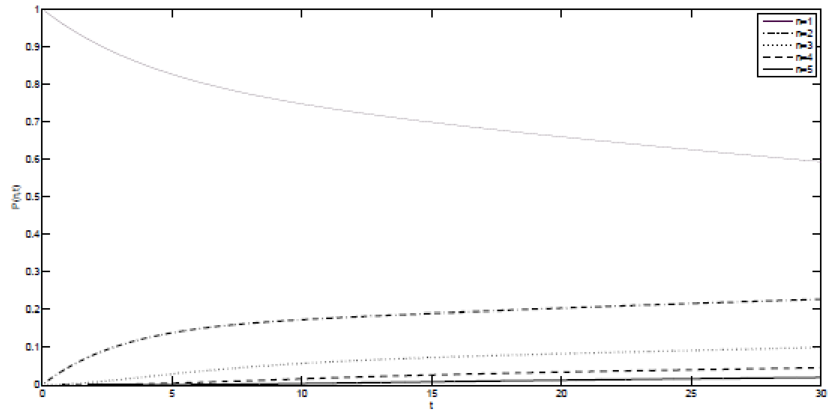
Counting process descriptors



Probabilities $P(N(t) = n)$ for $n \in \mathbb{N}$ and $t > 0$.



Expected number of failures at time t.



Probabilities $P(N(t) = n)$ for $n = 1, 2, 3, 4, 5$ and $t > 0$.

Conclusions & Extensions

- The failure times are considered to be dependent and not identically distributed, an assumption which is realistic in practice.
- The canonical representation of the non-stationary version of the MAP_2 is considered to model the failure times.
- We present a moments matching method estimation procedure to fit the non-stationary second-order MAP to sequences of operational times of N electrical components that are structurally equal.
- From the estimated parameters of the model, a number of key performance measures regarding the counting process, as the probability of N failures or the expected number of failures at time t, are inferred.

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